Wave-Obstacle interaction

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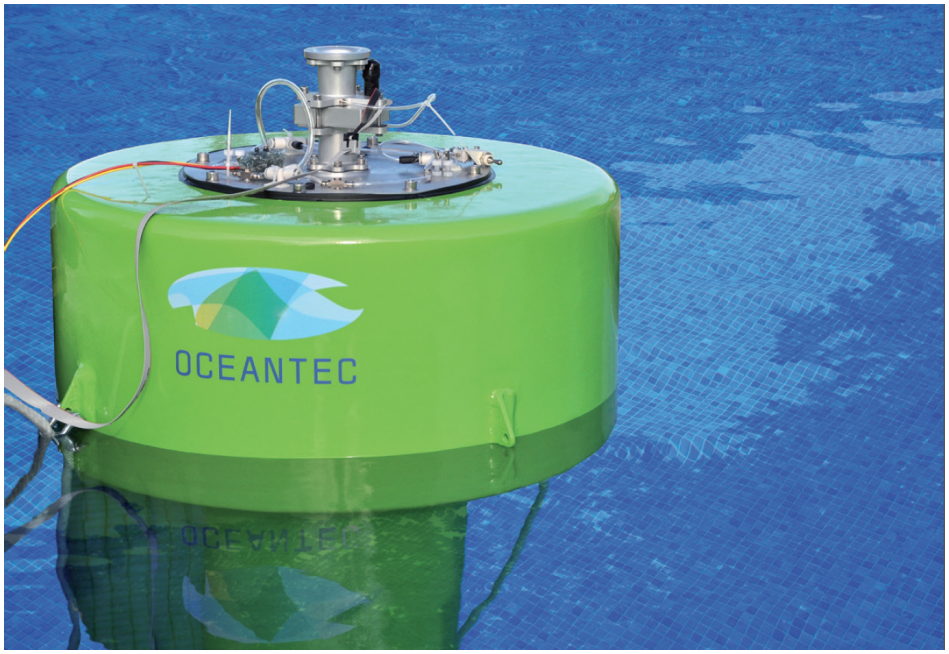
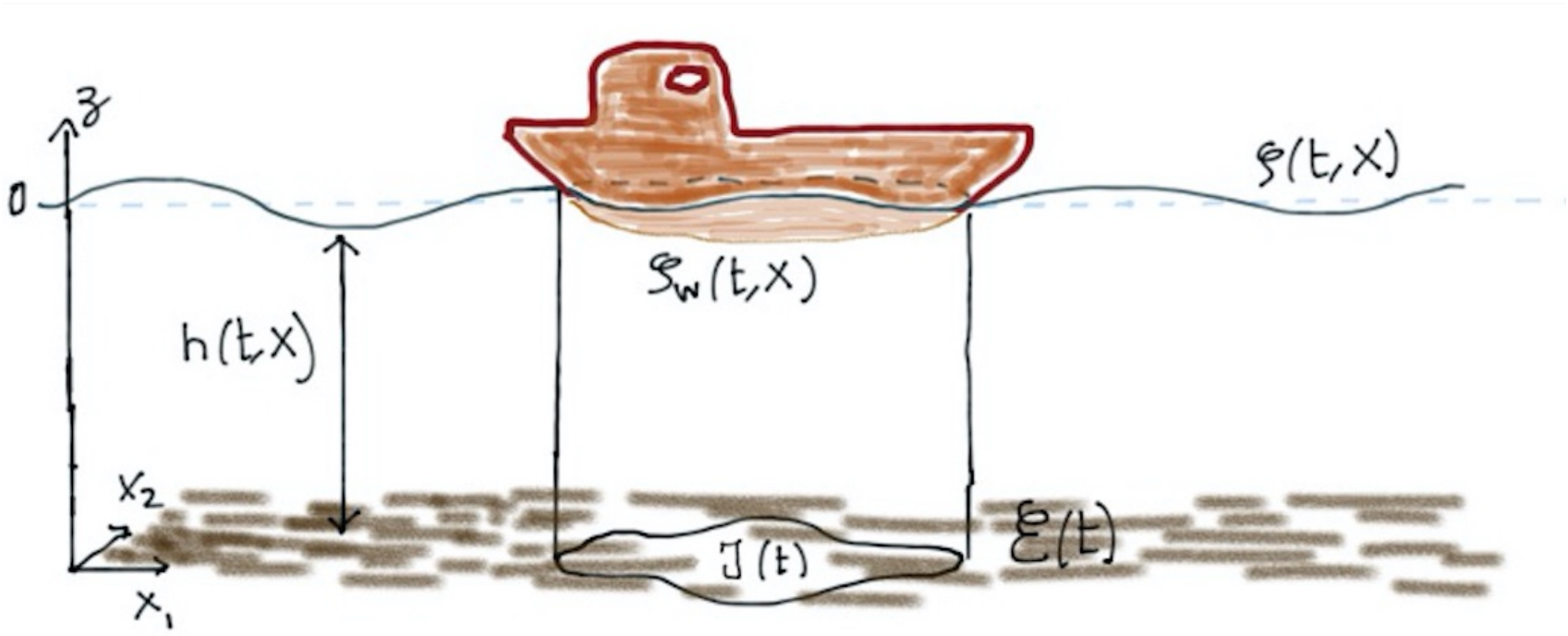
Motivations



⇝ The Hydrodynamics of Wave Energy Convertors,
Bordeaux, 18-20 juin 2019
`https://hywec2.sciencesconf.org/`
⇝ Simulation et Optimisation pour les Energies Marines Renouvelables,
Roscoff 2-5 juillet 2019,
`https://emrsim2019.sciencesconf.org/`

Notations



Floating device: ship or wave energy convertor

Notation

If f is defined on $\mathbb{R}^d$, we write

$$f_{\mathrm{e}} = f_{|\mathcal{E}} \quad \text{and} \quad f_{\mathrm{i}} = f_{|\mathcal{I}}$$

Basic equations

In the fluid domain Ω_t

$$\begin{aligned} \partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{X,z} \mathbf{U} &= -\frac{1}{\rho} \nabla_{X,z} P - g \mathbf{e}_z \\ \text{div } \mathbf{U} &= 0, \\ \text{curl } \mathbf{U} &= 0 \end{aligned}$$

At the surface

$$\boxed{\forall X \in \mathcal{E}(t),} \qquad \underline{P}_{\mathrm{e}}(t, X) = P_{\mathrm{atm}},$$

$$\forall X \in \mathbb{R}^d, \qquad \partial_t \zeta - \underline{U} \cdot N = 0 \qquad \text{with } N = \begin{pmatrix} -\nabla \zeta \\ 1 \end{pmatrix},$$

At the bottom

$$U_b \cdot N_b = 0 \qquad \text{with } N_b = \begin{pmatrix} -\nabla b \\ 1 \end{pmatrix}.$$

Interior equations and coupling

Constraint in the interior domain

The surface of the fluid coincides with the wetted portion of the body

$$\zeta_{\mathrm{i}} = \zeta_{\mathrm{w}}$$

Coupling conditions on $\Gamma(t) := \partial\mathcal{I}(t) = \partial\mathcal{E}(t)$

Continuity of the surface elevation and of the surface pressure

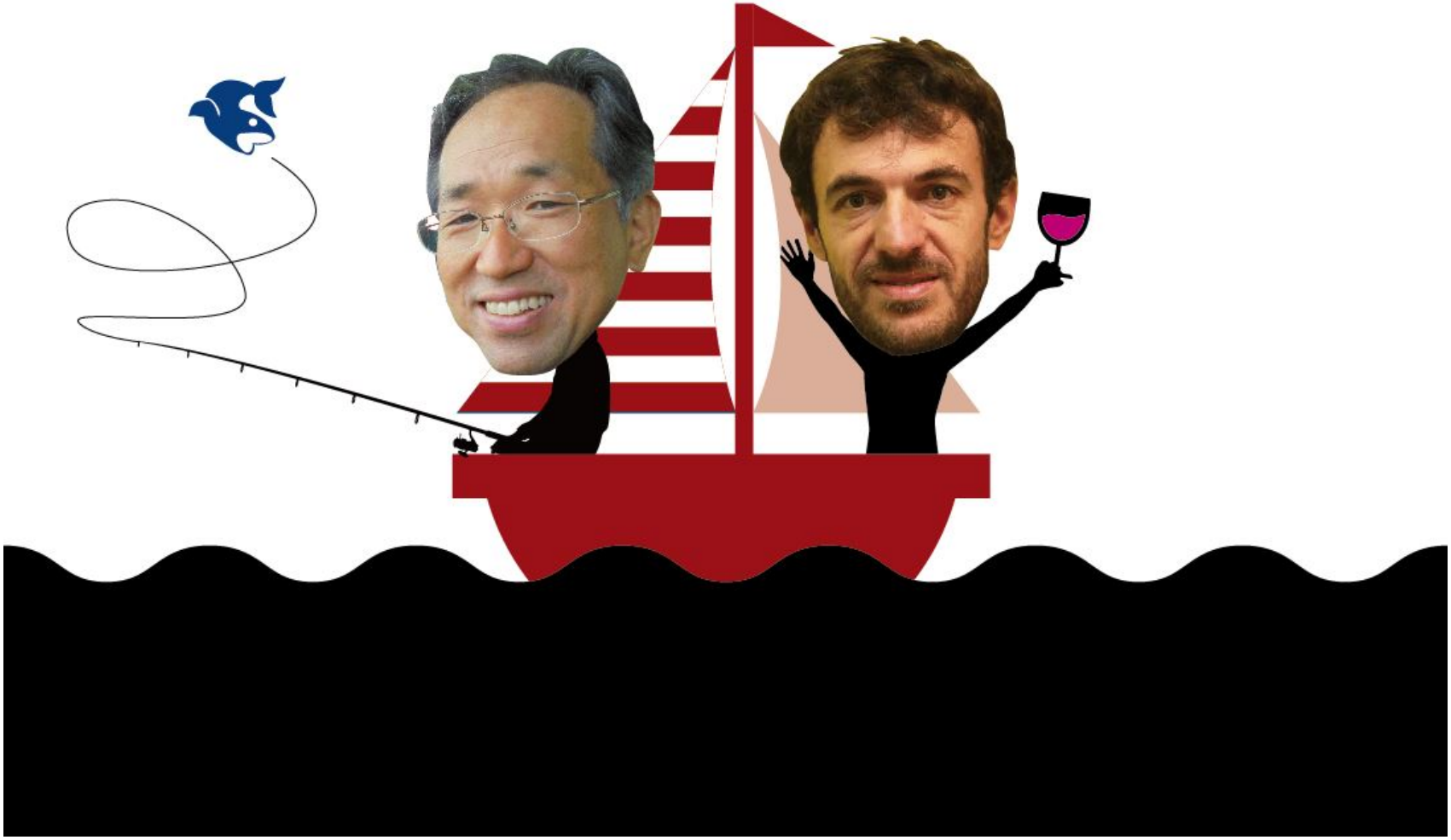
$$\zeta_{\mathrm{e}}(t,\cdot) = \zeta_{\mathrm{i}}(t,\cdot) \quad \text{and} \quad \underline{P}_{\mathrm{e}}(t,\cdot) = \underline{P}_{\mathrm{i}}(t,\cdot) \quad \text{on} \quad \Gamma(t)$$

Coupling with the solid equations: Newton's equations

$$\begin{cases} m\dot{U}_G & = -mg\mathbf{e}z + \int_{I(t)}(P_{\mathrm{i}} - P_{\mathrm{atm}})N_{\mathrm{w}}, \\ \dfrac{d}{dt}(\mathcal{I}\omega) & = \int_{I(t)}(P_{\mathrm{i}} - P_{\mathrm{atm}})\mathbf{r}_G \times N_{\mathrm{w}}. \end{cases}$$

$\rightsquigarrow$ (Partial) analysis of this problem: D. L. , Ann. of PDE, 2017

The one dimensional shallow water equations



Joint work with T. Iguchi

The equations

In the exterior domain

$$\begin{cases} \partial_t H + \partial_x Q = 0, \\ \partial_t Q + \partial_x(\frac{1}{H}Q^2 + \frac{1}{2}gH^2) = \frac{1}{\rho}H\partial_x \underline{P}_{\mathrm{atm}} = 0 \end{cases}$$

In the interior domain

$$\begin{cases} \partial_x Q_{\mathrm{i}} = -\partial_t H_{\mathrm{i}}, \\ \partial_t Q_{\mathrm{i}} + \partial_x(\frac{1}{H_{\mathrm{i}}}Q_{\mathrm{i}}^2 + \frac{1}{2}gH_{\mathrm{i}}^2) = -\frac{1}{\rho}H_{\mathrm{i}}\partial_x \underline{P}_{\mathrm{i}}. \end{cases}$$

Coupling conditions at $x = x_\pm(t)$

$$H(t,\cdot) = H_{\mathrm{i}}(t,\cdot), \qquad Q(t,\cdot) = Q_{\mathrm{i}}(t,\cdot), \quad \text{and} \quad \underline{P}_{\mathrm{atm}}(t,\cdot) = \underline{P}_{\mathrm{i}}(t,\cdot).$$

Coupling with the solid equations: the case of a fixed solid

$$\partial_t H_{\mathrm{i}} = 0 \quad \rightsquigarrow \quad Q_{\mathrm{i}}(t,x) = q_{\mathrm{i}}(t)$$

Reduction of the problem

- Interior equations

$$\begin{cases} Q(t,x) = q_{\mathrm{i}}, \\ \partial_t Q + \partial_x(\frac{1}{H}Q^2 + \frac{1}{2}gH^2) = -\frac{1}{\rho}H\partial_x \underline{P}_{\mathrm{i}} \end{cases}$$

with $\underline{P}_{\mathrm{i}}(t, x_\pm(t)) = P_{\mathrm{atm}}$

$\rightsquigarrow$ Solvability condition for $\underline{P}_{\mathrm{i}}$,

$$\partial_t q_{\mathrm{i}} = F(q_{\mathrm{i}}, x_+(t), x_-(t))$$

- The problem is reduced to

$$\begin{cases} \partial_t H + \partial_x Q = 0, \\ \partial_t Q + \partial_x(\frac{1}{H}Q^2 + \frac{1}{2}gH^2) = 0 \end{cases} \quad \text{in} \quad E(t)$$

$$\begin{cases} Q(t, x_\pm(t)) = q_{\mathrm{i}}(t), \\ \partial_t q_{\mathrm{i}} = F(q_{\mathrm{i}}, x_-, x_+). \end{cases} \quad \text{(boundary condition)}$$

$$H(t, x_\pm(t)) = H_{\mathrm{i}}(t, x_\pm(t)) \quad \text{(free boundary equation)}$$

A new kind of hyperbolic free boundary problem

Consider a 2×2 quasilinear hyperbolic system

$$\begin{cases} \partial_t U + A(U)\partial_x U = 0 & \text{in } (\underline{x}(t), +\infty) \\ U = U_{\mathrm{i}} & \text{on } [0, T] \times \{x = \underline{x}(t), \\ U = U^{\mathrm{in}} & \text{on } \{t = 0\} \times \mathbb{R}_+ \end{cases}$$

with $\underline{x}(0) = 0$

$\rightsquigarrow$ $U = U_{\mathrm{i}}$ is a boundary condition AND a free boundary equation

Remark

For the stability of shocks for conservation laws

$$\partial_t U + \partial_x (f(U)) = 0$$

the Rankine-Hugoniot condition also plays this double role

$$\dot{\underline{x}}[\![U]\!] = [\![f(U)]\!]$$

Analysis of the boundary condition/free boundary equation

$$\boxed{U(t,\underline{x}(t)) = U_{\mathrm{i}}(t,\underline{x}(t))}$$

$\rightsquigarrow$ Time differentiation

$$\partial_t U + \dot{\underline{x}}\partial_x U = \partial_t U_{\mathrm{i}} + \dot{\underline{x}}\partial_x U_{\mathrm{i}}$$

- Boundary condition

$$(\partial_t U - \partial_t U_{\mathrm{i}}) \cdot (\partial_x U - \partial_x U_{\mathrm{i}})^{\perp} = 0$$

- Free boundary evolution

$$\dot{\underline{x}} = -\frac{(\partial_t U - \partial_t U_{\mathrm{i}}) \cdot (\partial_x U - \partial_x U_{\mathrm{i}})}{|\partial_x U - \partial_x U_{\mathrm{i}}|^2}$$

Differences with stability of shocks $\dot{\underline{x}}[\![U]\!] = [\![f(U)]\!]$

1. The boundary condition is fully nonlinear
2. The evolution equations has a derivative loss

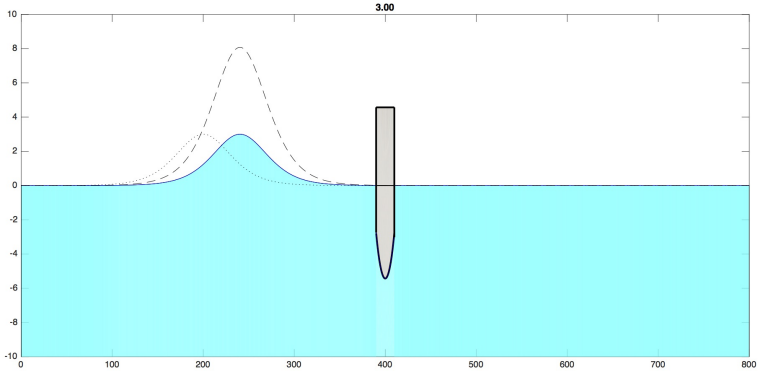
Main result

Theorem (T. Iguchi, D. L.)

Let $m \geq 2$, data compatible of order $m-1$. There exists a unique solution $\underline{x} \in H^m(0,T)$ and $U \in C([0,T]; H^m(\underline{x},\infty))$, with $U_{|x=\underline{x}} \in H^m(0,T)$.

- Fix the domain with a diffeomorphism $\varphi \rightsquigarrow u = U \circ \varphi$
  $\partial_t u + \mathcal{A}(u, \partial\varphi)\partial_x u = 0$ with $\partial_t \varphi = F(\partial u)$
- Kreiss symmetrizer $\rightsquigarrow$ Full regularity of the trace
  $\frac{1}{2}\partial_t(Su, u) + (M_0 u, u) - \left(S\mathcal{A}(u, \partial\varphi)u \cdot u\right)_{|x=0} = 0$
- L^2 estimates on $u_j = \partial_t^j u$
  $\partial_t u_j + \mathcal{A}(u, \partial\varphi)\partial_x u_j + B(u, \partial_x u)\partial\partial_t^j \varphi + C_{j+1} = l.o.t.$
  $\rightsquigarrow$ Needs a second order Alinhac's unknown
  - L^2-estimates on "$\partial_t^j u$" in terms of "$\partial_t^j u_{|t=0}$"
  - Control of "$\partial_t^j u_{|t=0}$" in H^{m-j} using the equation:
    $\partial_t U_{|t=0} = -A(U^{\text{in}})\partial_x U^{\text{in}}$, etc.
- Control of "space derivatives" in terms of "time derivatives":
  $\partial_x U = -A(U)^{-1}\partial_t U$, etc.

Including dispersive effects using a Boussinesq model



Fixed object with vertical sidewalls at $x_\pm = \pm R$

Joint work with Didier Bresch and Guy Métivier

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ (1 - \delta^2 \partial_x^2)\partial_t q + \varepsilon \partial_x (\frac{1}{h_0} q^2) + h \partial_x \zeta = -h \partial_x \underline{P} \end{cases}$$

with

$$\begin{cases} \underline{P} = P_{\mathrm{atm}} & \text{on } \mathcal{E} = (-\infty, -R) \cup (R, \infty), \\ \zeta = \zeta_{\mathrm{w}}(x) & \text{on } \mathcal{I} = (-R, R), \end{cases}$$

and one coupling condition

$$q(t, \pm R) = q_{\mathrm{i}}(t).$$

1. No continuity of the surface elevation $\rightsquigarrow$ no equation on $\underline{x}_\pm$ needed!
2. No continuity of the pressure $\rightsquigarrow$ an information is missing!

Boundary condition on the pressure

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ (1 - \delta^2 \partial_x^2)\partial_t q + \varepsilon \partial_x (\frac{1}{h_0} q^2) + h \partial_x \zeta = -h \partial_x \underline{P} \end{cases}$$

with $\zeta = \zeta_{\mathrm{w}}(x)$ on (x_-, x_+). For a an object with flat bottom,

$$\rightsquigarrow \quad \begin{cases} -\partial_x (h_{\mathrm{i}} \partial_x P_{\mathrm{i}}) = 0 & \text{in } (x_-, x_+), \\ P_{\mathrm{i}}(x_\pm) = P_{\mathrm{atm}} + P_{\mathrm{cor}}(x_\pm) \end{cases}$$

$$\alpha \dot{q}_{\mathrm{i}} = -[\![P_{\mathrm{i}}]\!] \qquad \alpha = \int_{x_-}^{x_+} \frac{1}{h_i}$$

$$= -[\![P_{\mathrm{cor}}]\!]$$

$\rightsquigarrow$ What is P_{cor} ???

Energy considerations

The Boussinesq system

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ (1 - \delta^2 \partial_x^2)\partial_t q + \varepsilon \partial_x (\frac{1}{h_0} q^2) + h \partial_x \zeta = -h \partial_x \underline{P} \end{cases}$$

has a local conservation law,

$$\partial_t \mathfrak{e} + \partial_x \mathfrak{F} = 0$$

with

$$\mathfrak{e} = \mathfrak{e}(\zeta, q, \partial_x q)$$
$$\mathfrak{F} = q \mathfrak{F}_0(\zeta, q, \partial_x \partial_t q)$$

$\rightsquigarrow$ We should have conservation of the total energy

$$E_{\mathrm{tot}} = \int_{(-\infty,-R)\cup(R,\infty)} \mathfrak{e} + \int_{(-R,R)} \mathfrak{e}$$

$$\mathfrak{e} = \tfrac{1}{2}\zeta^2 + \tfrac{\varepsilon}{6}\zeta^3 + \tfrac{1}{2h_0}q^2 + \tfrac{1}{2}\delta^2(\partial_x q)^2$$

$$\rightsquigarrow \qquad [\![\mathfrak{F}]\!] + \alpha q_{\mathrm{i}}\dot{q}_{\mathrm{i}} = 0.$$

But we have

1. $\mathfrak{F} = q\mathfrak{F}_0$
2. $q(t,\pm R) = q_{\mathrm{i}}(t)$
3. $\alpha\dot{q}_{\mathrm{i}} = -[\![P_{\mathrm{cor}}]\!]$

$$\rightsquigarrow \qquad -[\![P_{\mathrm{cor}}]\!] = -[\![\mathfrak{F}_0]\!]$$

We deduce

$$\boxed{-\alpha\dot{q}_{\mathrm{i}} = [\![\zeta + \varepsilon\frac{1}{2}\zeta^2 - \delta^2\partial_x\partial_t q]\!]}$$

.

A transmission problem

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ (1 - \delta^2 \partial_x^2)\partial_t q + \varepsilon \partial_x (\frac{1}{h_0} q^2) + h \partial_x \zeta = 0 \end{cases} \qquad \text{if } |x| > R$$

with transmission conditions

1. $[\![q]\!] = 0$
2. $-\delta^2 \partial_t [\![\partial_x q]\!] + [\![\zeta + \varepsilon \frac{1}{2} \zeta^2]\!] = -\alpha \dot{q}_{\mathrm{i}}$ with $q_{\mathrm{i}}(t) = q(t, x_\pm)$.

$\rightsquigarrow$ Change of variable $\theta = \zeta + \varepsilon \frac{1}{2} \zeta^2$,

$$\begin{cases} (1 + \varepsilon c(\theta))\partial_t \theta + \partial_x q = 0, \\ (1 - \delta^2 \partial_x^2)\partial_t q + \varepsilon \partial_x (\frac{1}{h_0} q^2) + \partial_x \theta = 0 \end{cases} \qquad \text{if } |x| > R$$

with transmission conditions

1. $[\![q]\!] = 0$
2. $-\delta^2 \partial_t [\![\partial_x q]\!] + [\![\theta]\!] = -\alpha \dot{q}_{\mathrm{i}}$

Remark

No result for the IBVP for Boussinesq systems.

Mathematical analysis: local existence

$$\begin{cases} (1+\varepsilon c'(\theta))\partial_t\theta + \partial_x q = 0, \\ (1-\delta^2\partial_x^2)\partial_t q + \partial_x(\theta+\varepsilon q^2) = 0 \end{cases} \qquad \text{if } |x| > R$$

with initial condition $(\theta^{\mathrm{in}}, q^{\mathrm{in}})$ and transmission conditions

1. $[\![q]\!] = 0$
2. $-\delta^2\partial_t[\![\partial_x q]\!] + [\![\theta]\!] = -\alpha\dot{q}_{\mathrm{i}}$

Proposition

For $n \geq 0$, let $(\theta^{\mathrm{in}}, q^{\mathrm{in}}) \in H^n \times H^{n+1}$ satisfy $[\![q^{\mathrm{in}}]\!] = 0$ and $\inf\{1+\varepsilon c'(\theta^{\mathrm{in}})\} > 0$. For $\varepsilon \in [0,1]$ and $\delta > 0$, there is $T > 0$ and a unique solution in $C^1([0,T[; H^n \times H^{n+1})$.
Moreover, one has the blow up criterion

$$\lim \sup_{T \to T^*} \left\| \theta, q, \partial_x q, 1/(1+\varepsilon c'(\theta)) \right\|_{L^\infty([0,T]\times\mathcal{E})} = +\infty. \tag{1}$$

Proof: Reduction to an ODE

$$\begin{cases} (1+\varepsilon c'(\theta))\partial_t\theta + \partial_x q = 0, \\ (1-\delta^2\partial_x^2)\partial_t q + \underbrace{\partial_x(\theta+\varepsilon q^2)}_{:=\Gamma} = 0 \end{cases} \qquad \text{if } |x| > R$$

1. $[\![q]\!] = 0$
2. $-\delta^2\partial_t[\![\partial_x q]\!] + [\![\theta]\!] = -\alpha\dot{q}_{\mathrm{i}}$

- Denote R_0 the inverse of $(1-\delta^2\partial_x^2)$ with Dirichlet BC at $x = \pm R$,

$$\partial_t q = -R_0\Gamma + \dot{q}^{\mathrm{i}}\exp(-\frac{1}{\delta}|x|_R)$$

- $\partial_t\partial_x q(\pm R) = -\partial_x R_0\Gamma \mp \frac{1}{\delta}\dot{q}^{\mathrm{i}}$
- $\partial_t[\![\partial_x q]\!] = -[\![\partial_x R_0\Gamma]\!] - \frac{2}{\delta}\dot{q}^{\mathrm{i}}$
- Use second transmission condition

$$\dot{q}^{\mathrm{i}} = \frac{1}{\alpha+2\delta}\left(\delta^2[\![-\partial_x R_0\Gamma]\!] - [\![\theta]\!]\right)$$

- We have an ODE on $H^n \times H^{n+1}$ and Cauchy-Lipschitz.

Hyperbolic versus dispersive IBVP

Dispersive

$$\partial_t U = \mathcal{L}(U) := \begin{pmatrix} -\Phi \\ -\mathcal{R}(\Gamma, [\![\theta]\!]) \end{pmatrix} \qquad U = (\theta, q)$$

with $\Phi = \frac{1}{1+\varepsilon c'(\theta)} \partial_x q, \qquad \Gamma = \partial_x(\theta + \varepsilon q^2).$

and $\mathcal{R}(\Gamma, \rho) = R_0 \Gamma + \frac{1}{\alpha + 2\delta} \left(\delta^2 [\![\partial_x R_0 \Gamma]\!] + \rho \right) e^{-\frac{1}{\delta}|x|_R}.$

- $\rightsquigarrow$ ODE
- $\rightsquigarrow$ Smooth solution without compatibility conditions
- $\rightsquigarrow$ BC $[\![q]\!] = 0$ and $-\delta^2 \partial_t [\![\partial_x q]\!] + [\![\theta]\!] = -\alpha \dot{q}_{\mathrm{i}}$ propagated by the eqs

Hyperbolic

$$\partial_t U = \mathcal{L}_0(U) := \begin{pmatrix} -\Phi \\ -\Gamma \end{pmatrix}$$

- $\rightsquigarrow$ Opposite behavior !
- $\rightsquigarrow$ Role of the Dispersive Boundary Layer

Energy estimates

$$\begin{cases} (1+\varepsilon c(\underline{\theta}))\partial_t\theta + \partial_x q = 0, \\ (1-\delta^2\partial_x^2)\partial_t q + 2\varepsilon \underline{q}\partial_x(q) + \partial_x\theta = 0 \end{cases} \qquad \text{if } |x| > R$$

1. $[\![q]\!] = 0$
2. $-\delta^2\partial_t[\![\partial_x q]\!] + [\![\theta]\!] = -\alpha\dot{q}_{\mathrm{i}}$

GOAL

Uniform energy estimates with respect to δ over a time $O(1/\varepsilon)$

$$E^{\mathrm{tot}} = \int_{\mathcal{E}} \left[\frac{1}{2}(1+c'(\underline{\theta}))\theta^2 + \frac{1}{2}q^2 + \frac{1}{2}\delta^2(\partial_x q)^2\right] + \frac{1}{2}\alpha(q^{\mathrm{i}})^2$$

Proposition

If $\underline{U} = (\underline{\theta}, \underline{q}) \in W^{1,\infty}$ *and* $(1+\varepsilon c'(\underline{\theta})) \geq c_0 > 0$, *then*

$$E^{\mathrm{tot}}(U)(t) \leq e^{\varepsilon\gamma t} E^{\mathrm{tot}}(U^{\mathrm{in}})$$

Higher order energy estimates

$$\begin{cases}(1+\varepsilon c((\underline{\theta}))\partial_t\theta+\partial_x q=0,\\ (1-\delta^2\partial_x^2)\partial_t q+2\varepsilon\underline{q}\partial_x(q)+\partial_x\theta=0\end{cases}\qquad \text{if } |x|>R$$

1. $[\![q]\!]=0$
2. $-\delta^2\partial_t[\![\partial_x q]\!]+[\![\theta]\!]=-\alpha\dot{q}_{\rm i}$

- L^2-energy estimates
- L^2-energy estimates on the time derivatives $U_j=\partial_t^j U$
  Problem 1 How to control $U_{j|_{t=0}}$ in terms of $U^{\rm in}$?

$$\partial_t q=-R_0\partial_x\big(\theta+\varepsilon q^2\big)+\dot{q}^{\rm i}\exp(-\frac{1}{\delta}|x|_R)$$

  $\rightsquigarrow$ Control at $t=0$: Both terms are singular in H^n as $\delta\to 0$
  $\rightsquigarrow$ Need compatibility conditions to control the dispersive boundary layer
- Use eq. to control space derivatives in terms of time derivatives
  $\rightsquigarrow$ $\partial_x q$ OK
  Problem 2 How to control $\partial_x\theta$?

Main result

Theorem (D. Bresch, D. L., G. Métivier)

Initial data in H^m (m large enough)+ compatibility conditions.
Then there exists $T > 0$ such that for all $0 < \varepsilon < 1$ and $\delta^2 = O(\varepsilon)$, there is a unique solution $U \in C^k([0, \frac{T}{\varepsilon}]; H^{m-k})$ with uniform energy estimate.

Remark

The compatbility conditions degenerate into the hyperbolic compatibility conditions if $\delta = 0$.

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Thanks for your attention