Défis mathématiques pour la modélisation des écoulements de débris

G. Narbona-Reina, in collaboration with
E.D. Fernández-Nieto, F. Bouchut, A. Mangeney

50 ans Laboratoire Jacques-Louis Lions
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Outline

Outline

What is a debris flow?

Debris flows

Debris flow is composed by a mixture of solid particles mixture with fluid (water, loose mud, ...) that rush down a slope (mountainside).

Hazard: Some of them are very fast and produce serious damages in nearby settlements.

▶ A good approximation of the debris deposit and its evolution could help to mitigate related damages through risk evaluation.

Debris flow in Illgraben (Switzerland), 2016

Debris flow in Brasil, 2011

Outline

Debris flows modelling

Important ingredients to take into account

- 3D behavior
  vertical estructure of the flow
- Different size particles
  debris flow is made up of sand-size or larger particles
- Two-phase nature: solid-fluide
  in debris flows particles move independently within the flow ▶ two velocities are required
- Rheology
  viscous or inertial grain-grain interactions: Coulomb, Bingham, dilatant, …

Rheology

Example for **dry granular**: $\mu(I)$-rheology,

Jop, P., Forterre, Y., Pouliquen, O. A constitutive law for dense granular flows. Nature 441 (7094), 2006.

Granular viscosity $\eta = \dfrac{\mu(I)p}{\dot{\gamma}}$ where

$$\mu(I) = \mu_s + \frac{\mu_d - \mu_s}{1 + \dfrac{I_0}{I}}; \qquad I = \frac{d_g \dot{\gamma}}{\sqrt{p/\rho_g}}$$

I is the inertial number (depending on the flow regime) and $\dot{\gamma}$ strain rate.

◦ When used with incompressible Navier-Stokes equations. . .

- Well posed just for intermediate values of I (dense inertial regime)
- Ill posed for very low or very high values of I (quasi-static or collisional regime)

  T. Barker, D.G. Schaeffer, P. Bohorquez, J.M.N.T. Gray, Well-posed and ill-posed behaviour of the mu(I)-rheology for granular flow, J. Fluid Mech. 779, 2015.

- Viscosity is not defined when $\dot{\gamma} = 0$.

  The easiest way to solve it: regularization $\eta = \dfrac{\mu(I)p}{\sqrt{\dot{\gamma}^2 + \delta^2}}$

Rheology

For **fluidized granular**

M. Trulsson, B. Andreotti, and P. Claudin. Transition from the viscous to inertial regime in dense suspensions. Phys. Rev. Lett. 109, 118305, 2012.

The inertial number depends on the flow regime:

$$\text{inertial:} \quad I_i = \frac{d_g \dot{\gamma}}{\sqrt{p_g/\rho_g}}; \qquad \text{viscous:} \quad I_v = \frac{\eta_f \dot{\gamma}}{p_g}.$$

The proposal:

$$\tilde{I} = I_v + \alpha I_i^2 \qquad (\alpha \text{ a constant})$$

and

$$\mu(I) = \mu_c + \frac{\mu_F - \mu_c}{1 + \sqrt{I_0/\tilde{I}}}$$

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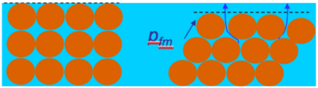
- Rheology
  viscous or inertial grain-grain interactions: Coulomb, Bingham, dilatant, …
- Dilatancy
  affects the dynamics of the debris flows and it is related to the pore fluid pressure

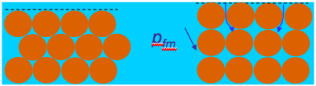
Dilatancy

φ: solid volume fraction with critical state $\varphi_{eq} = \varphi_{eq}(p)$ (not constant)

When deformed ($\dot{\gamma} \neq 0$)

$\varphi < \varphi_{eq}$ (loose)	$\varphi > \varphi_{eq}$ (dense)
▼	▼
Granular medium contracts	Granular medium dilates
▼	▼
Fluid is expelled	Fluid is sucked
▼	▼
Pore pressure increases	Pore pressure decreases
▼	▼
Liquefaction	Stiffening of the granular matrix





Debris flows modelling

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- Dilatancy
  affects the dynamics of the debris flows and it is related to the pore fluid pressure
- Bottom variation

It is hard to tackle all items!

Common simplifications in mathematical models

- 3D resolution $\rightsquigarrow$ Depth-averaged models
- multi-species model $\rightsquigarrow$ One single particle sort
- proper two-phase model $\rightsquigarrow$ Mixture theory: one velocity
- complex rheology $\rightsquigarrow$ Simple or no viscous effects
- dilatancy law related to rheology $\rightsquigarrow$ No dilatancy effects
- bottom variation $\rightsquigarrow$ Local coordinates

Good mathematical properties for the model to be physically relevant

- Dissipative energy balance.
- Mass and momentum conservation.

Outline

Debris flows models

How is it tackled in the literature?

- 3D resolution $\rightsquigarrow$ $\boxed{\text{Depth-averaged models}}$
- multi-species model $\rightsquigarrow$ $\boxed{\text{One single particle sort}}$
- proper two-phase model $\rightsquigarrow$ Mixture theory: one velocity
- complex rheology $\rightsquigarrow$ $\boxed{\text{Mohr-Coulomb friction theory}}$
- dilatancy law related to rheology $\rightsquigarrow$ No dilatancy effects
- bottom variation $\rightsquigarrow$ $\boxed{\text{Local coordinates}}$

How is it tackled in the literature?

Mixture theory

Iverson (1997) proposed a model to study shallow partially fluidized avalanches, a mixture of a granular material and a fluid.

R.M. Iverson. The physics of debris flows. Rev. Geophys., 35:245-296, 1997.

- Assumptions:
  - one velocity for the mixture (no relative motion)
  - rheology: viscosity and Coulomb friction theory
  - no dilatancy
- Ingredients:
  - stress tensors are defined as contributions from each phase
  - a pore pressure advection-diffusion equation is added based on experimental measurements

$$\partial_t h + \partial_x (hu) = 0$$

$$\begin{aligned}
&\rho(\partial_t (hu) + \partial_x (hu^2)) = \rho g h - h\partial_x p_{bed} - sgn(u)(\rho g h - p_{bed}) \tan\delta \\
&\qquad -\nu_f \mu \dot{\gamma} - h k_{act/pass} \partial_x (\rho g h - p_{bed}) \\
&\partial_t p_{bed} + u\partial_x p_{bed} = D(\partial_z^2 p)_{bed}
\end{aligned}$$

Mixture theory

- they add dilatancy effects in 2014

  R. M. Iverson and D. L. George. A depth-averaged debris-flow model that includes the effects of evolving dilatancy. I. Physical basis. Proc. R. Soc. A, 470, 2014.

- An evolution equation for the fluid pore pressure is established using an empirical dilatancy law and a Darcy law.

Pore pressure: $p_f = p^H + p^E$

$$\nabla \cdot v = \dot{\gamma} \tan\psi - \alpha \partial_t (\sigma - p_f)$$

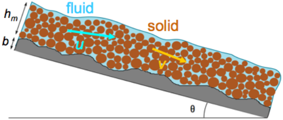
$$(1 - \varphi)(u - v) = -\frac{\kappa}{\eta_f} \partial_x p^E$$

∘ ψ: dilatancy angle depending on φ

Two-phase theory

Based on the two-phase (solid-fluid) theory, debris flows are usually described by the conservation of mass and momentum for each phase.

T.B. Anderson, R. Jackson. A fluid mechanical description of fluidized beds. Ind. Eng. Chem. Fundam. 6, 1967.



$$
\begin{aligned}
&\partial_t(\rho_s\varphi) + \nabla\cdot(\rho_s\varphi v) = 0\\
&\partial_t(\rho_f(1-\varphi)) + \nabla\cdot(\rho_f(1-\varphi)u) = 0\\
&\rho_s\varphi(\partial_t v + (v\cdot\nabla)v) = -\nabla\cdot T_s + f_0 + \rho_s\varphi g\\
&\rho_f(1-\varphi)(\partial_t u + (u\cdot\nabla)u) = -\nabla p_f - f_0 + \rho_f(1-\varphi)g
\end{aligned}
$$

Buoyancy and drag frictional force between phases:

$$f_0 = -\varphi\nabla p_f + Fric$$

In 1D this system has 4 equations and 5 unknowns: solid volume fraction, pressures and velocities for both phases.

- An additional equation is necessary to close the system.

Two-phase theory

Pitman and Le (2005) and Pelanti et al. (2008)

E.B. Pitman, L. Le. A two-fluid model for avalanche and debris flows. Phil. Trans. R. Soc. A 363, 2005.

M. Pelanti, F. Bouchut, A. Mangeney. A Roe-type scheme for two-phase shallow granular flows over variable topography. M2AN 42, 2008.

- Assumptions:
  - rheology: Coulomb friction theory
  - no dilatancy
  - two boundary conditions are imposed at the free surface: $p_s = p_f = 0 \Rightarrow$ overdeterminated problem at the free surface
- Results:
  - no closure equation to Jackson's model
  - no energy balance

Two-phase theory

Our first contribution. . .

F. Bouchut, E.D. Fernández-Nieto, A. Mangeney, G. Narbona. A two-phase shallow debris flow model with energy balance. ESAIM: Mathematical Modelling and Numerical Analysis, vol. 49, 2014

- Assumptions:
  - rheology: Coulomb friction theory
  - no dilatancy
  - Closure equation for solid incompressibility: $\boxed{\nabla \cdot v = 0}$
  - Only one boundary condition is imposed on the free surface:

$$(p_s + p_f)N_x = 0$$

- Results:
  - well-posed initial governing system
  - dissipative energy balance
  - An extra unknown is proposed ($p_{f\,|bed}$), that is the Lagrange multiplier associated to the closure

The model:

- Mass equations

$$\partial_t(h\varphi) + \partial_x(h\varphi v) = 0$$
$$\partial_t(h(1-\varphi)) + \partial_x(h(1-\varphi)u) = 0$$

- Momentum equations:

$$\begin{aligned}\rho_s\varphi(\partial_t v + v\partial_x v) = {} & (1-\varphi)\partial_x(p_f)_{bed} - (1-\varphi)\rho_f g\cos\theta\partial_x h \\ & -\varphi\rho_s g\partial_x(b+h) - \tfrac{1}{2}(\rho_s-\rho_f)gh\cos\theta\partial_x\varphi \\ & -\varphi\rho_s g\sin\theta + \beta(u-v) - sign(v)\tan\delta\varphi(\rho_s-\rho_f)g\cos\theta\end{aligned}$$

$$\begin{aligned}\rho_f(1-\varphi)(\partial_t u + u\partial_x u) = {} & -(1-\varphi)\partial_x(p_f)_{bed} - (1-\varphi)\rho_f g\cos\theta\partial_x b \\ & -(1-\varphi)\rho_f g\sin\theta - \beta(u-v)\end{aligned}$$

- Closure equation:

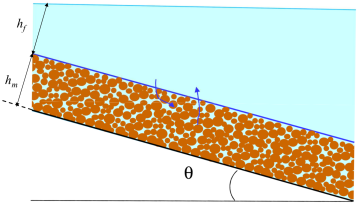
$$\partial_x(h(1-\varphi)(u-v)) = 0$$

Two-phase theory

Pailha and Pouliquen (2009)

M. Pailha, O. Pouliquen: A two-phase flow description of the initiation of underwater granular avalanches. J. Fluid Mech. vol. 633, 2009.

- rheology: Coulomb friction theory
- only for immersed granular flows



Two-phase theory

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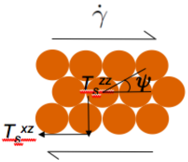
- only for immersed granular flows
- Closure equation related to dilatancy: $\boxed{\nabla \cdot v = \dot{\gamma} \tan \psi}$
  - ψ: dilatancy angle depending on φ
  - linearization: $\boxed{\tan \psi = K(\varphi - \varphi_{eq})}$

S. Roux and F. Radjai, Texture-dependent rigid plastic behavior. In Proc. of Physics of Dry Granular Media 1997.



if $\varphi < \varphi_{eq}$: compression

$$\nabla \cdot v < 0$$

if $\varphi > \varphi_{eq}$: dilation

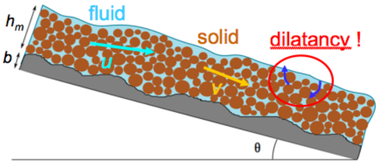
$$\nabla \cdot v > 0$$

Two-phase theory

Our second contribution...

Bouchut, F., Fernández-Nieto, E. D., Mangeney, A., and Narbona-Reina, G. A two-phase two-layer model for fluidized granular flows with dilatancy effects, J. FLUID MECH., 801, 2016.

- Closure equation related to dilatancy: $\boxed{\nabla \cdot v = \dot{\gamma} \tan \psi = K\dot{\gamma}(\varphi - \varphi_{eq}) \equiv \Phi}$



dilatancy scheme

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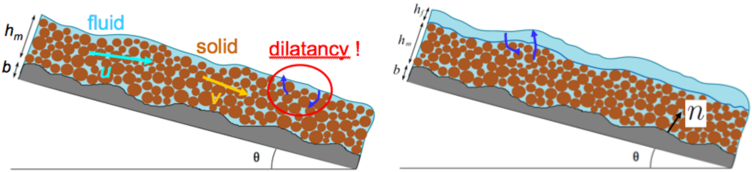

- we add a thin only-fluid layer on the top
- two-phase two-layer model: boundary conditions at the interface!

Boundary conditions at the interface

- A kinematic condition for the solid phase,

$$\tilde{N}_t + v \cdot \tilde{N}_X = 0.$$

(where we denote by $\tilde{N} = (\tilde{N}_t, \tilde{N}_X)$ a time-space upward normal to the interface).

- A Navier fluid friction condition

$$\left(\frac{T_{fm} + T_f}{2} \tilde{N}_X \right)_\tau = -k_i (u_f - u_m)_\tau .$$

where $k_i \geq 0$ is a friction coefficient.

- Energy balance through the interface ($\rightsquigarrow p_s = 0$).

$$T_s \tilde{N}_X = \left(\frac{\rho_f}{2} \left((u_m - u_f) \cdot \frac{\tilde{N}_X}{|\tilde{N}_X|} \right)^2 + \left((T_{fm} \tilde{N}_X) \cdot \frac{\tilde{N}_X}{|\tilde{N}_X|^2} - p_{fm} \right) \frac{\varphi^*}{1 - \varphi^*} \right) \tilde{N}_X .$$

Boundary conditions at the interface

- **Additional jump relations have to be prescribed**.
  Rankine-Hugoniot conditions for the exchange rate of fluid phase.

$$\tilde{N}_t + u_f \cdot \tilde{N}_X = (1 - \varphi^*)(\tilde{N}_t + u_m \cdot \tilde{N}_X) \equiv \mathcal{V}_f,$$

  where:
  - φ^* is the value of the solid volume fraction at the interface.
  - The term $\mathcal{V}_f$ defines the fluid mass that is transferred from the mixture to the fluid-only layer.
    ◦ $\mathcal{V}_f < 0$ means that the fluid is transferred from the fluid-only region to the mixture region.

Asymptotic hypothesis ($\epsilon = H/L \ll 1$)
The drag term is defined by

$$f = \beta(u_m - v); \qquad \beta = (1-\varphi)^2 \frac{\eta_f}{\kappa},$$

where η_f is the dynamic viscosity of the fluid and κ is the hydraulic permeability

We shall consider two possible sets of assumptions.

1. The drag term is strong: $\boxed{\beta \sim \epsilon^{-1}}$
   Since the drag force $\tilde{\beta}(u_m - v)$ has to balance gravity terms, it necessarily remains bounded. This implies that

$$u_m^x - v^x = O(\epsilon).$$

2. The drag term is moderate: $\boxed{\beta = O(1)}$
   In this case one has just $u_m^x - v^x = O(1)$.

Debris flows models

- Fluid and solid pressures:

$$p_f = \rho_f g \cos\theta (b + h_m + h_f - z) + p_f^e;$$

$$p_s = \varphi(\rho_g - \rho_f) g \cos\theta (b + h_m - z) - p_f^e; \qquad p_f^e = \frac{\beta}{1-\varphi} \int_z^{b+h_m} (u_m^z - v^z) dz$$

We have thus to evaluate $u_m^z - v^z$ up to $O(\epsilon^2)$ errors.

  - Using the closure equation ($\nabla \cdot v = \Phi$), mass equations and boundary conditions:

$$u_m^z - v^z = (u_m^x - v^x)\partial_x b - \frac{z-b}{1-\varphi}\Big(\Phi + \nabla_{\mathbf{x}} \cdot \big((1-\varphi)(u_m^x - v^x)\big)\Big) + O(\epsilon^3).$$

- Fluid and solid pressures:
  - In the model: $(p^e_{fm})_{|b}, \overline{p}^e_{fm}$
    1. $\beta = O(\epsilon^{-1})$

$$(p^e_{fm})_{|b} = -\frac{\bar{\beta}}{(1-\varphi)^2}\frac{h_m^2}{2}\Phi + \mathcal{O}(\epsilon^2), \quad \overline{p}^e_{fm} = -\frac{\bar{\beta}}{(1-\varphi)^2}\frac{h_m^2}{3}\Phi + O(\epsilon^2).$$

    2. $\beta = O(1)$

$$(p^e_{fm})_{|b} = \frac{\bar{\beta}}{1-\varphi}\left(h_m(u^x_m - v^x)\partial_x b - \frac{h_m^2}{2(1-\varphi)}\left(\Phi + \partial_x\left((1-\varphi)(u^x_m - v^x)\right)\right)\right) + O(\epsilon^3),$$

$$\overline{p}^e_{fm} = \frac{\bar{\beta}}{1-\varphi}\left(\frac{h_m}{2}(u^x_m - v^x)\partial_x b - \frac{h_m^2}{3(1-\varphi)}\left(\Phi + \partial_x\left((1-\varphi)(u^x_m - v^x)\right)\right)\right) + O(\epsilon^3).$$

Debris flows models

The model:

∘ Mass equations

$$
\begin{aligned}
&\partial_t(h_m\varphi) + \partial_x(h_m\varphi v) = 0\\
&\partial_t(h_m(1-\varphi)) + \partial_x(h_m(1-\varphi)u) = -\mathcal{V}_f\\
&\partial_t(h_f) + \partial_x(h_f u_f) = \mathcal{V}_f
\end{aligned}
$$

∘ Momentum equations:

$$
\begin{aligned}
\rho_s\varphi(\partial_t v + v\partial_x v) = {} & (1-\varphi)\partial_x(p_{fm})_{bed} - \varphi g\cos\theta(\rho_s\partial_x(b+h_m) + \rho_f\partial_x h_f)\\
& -\tfrac{1}{2}(\rho_s-\rho_f)gh\cos\theta\partial_x\varphi\\
& -\varphi\rho_s g\sin\theta + \beta(u-v) - sign(v)\tan\delta_{eff}\tfrac{1}{h_m}(\varphi(\rho_s-\rho_f)g\cos\theta h_m - (p_{fm})_{bed})_+
\end{aligned}
$$

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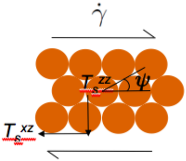
\begin{aligned}
\rho_f(1-\varphi)(\partial_t u + u\partial_x u) = {} & -(1-\varphi)\partial_x(p_{fm})_{bed} - (1-\varphi)\rho_f g\cos\theta\partial_x(b+h_m+h_f)\\
& -(1-\varphi)\rho_f g\sin\theta - \beta(u-v) - \tfrac{1}{h_m}\Big((\tfrac{1}{2}\rho_f\mathcal{V}_f - k_i)(u_f-u) + k_b u\Big)
\end{aligned}
$$

$$
\rho_f(\partial_t u_f + u\partial_x u) = -\rho_f g\cos\theta\partial_x(b+h_m+h_f) - \rho_f g\sin\theta - \tfrac{1}{h_f}\Big((\tfrac{1}{2}\rho_f\mathcal{V}_f + k_i)(u_f-u)\Big)
$$

∘ Closure equation:

$$
\partial_t\varphi + v\partial_x\varphi = -\varphi\Phi
$$

Debris flows models



if $\varphi < \varphi_{eq}$: compression

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if $\varphi > \varphi_{eq}$: dilation

$$\nabla \cdot v > 0$$

- $\bar{p}_{fm} = \rho_f g \cos\theta(b + h_m + h_f) + \bar{p}^e_{fm}$; $\quad p^e_{fm} = -\dfrac{\bar{\beta}}{(1-\varphi)^2}\dfrac{h_m^2}{3}K\dot{\gamma}(\varphi - \varphi_{eq})$

dilatancy scheme

- Transference fluide:

$$\mathcal{V}_f = -h_m\Phi - \partial_x((1-\varphi)h_m(u-v))$$

- Impact of the dilatancy angle on the Coulomb friction force:

$$T_s^{xz} = -\tan(\underbrace{\delta + \psi}_{\delta_{eff}})\operatorname{sign}(v)\, p_{s|bed}$$

$$p_{s|bed} = \varphi(\rho_s - \rho_f)gh_m\cos\theta - (p^e_f)_{bed}; \qquad (p^e_f)_{bed} = -\frac{\beta}{(1-\varphi)^2}\frac{h_m^2}{2}\Phi$$

Outline

Inmersed non-uniform test. Comparison with experimental data

- We perform the tests of the collapse of a granular column in a viscous liquid that is experimentally investigated in

  L. Rondon, O. Pouliquen and P. Aussillous, *Granular collapse in a fluid: Role of the initial volume fraction*, Physics of Fluids, **23**, 073301, 2011

- This experiment corresponds to a one-dimensional dam-break with an initial rectangular mixture mass of height H_i and width L_i that is at rest with a volume fraction φ_i.
- The experiment exhibits results for a loose and a dense initial packing configurations characterized by the data ($\varphi_i = 0.55, L_i = 6$ cm, $H_i = 4.8$ cm) and ($\varphi_i = 0.6, L_i = 6$ cm, $H_i = 4.2$ cm) respectively.
- $\rho_s = 2500$ kg m^{-3} $d = 225\ \mu$m, $\rho_f = 1000$ kg m^{-3} and $\eta = 12 \times 10^{-3}$ Pa.s.

Bouchut, F., Fernández-Nieto, E. D., Koné, E. H., Mangeney, A., and Narbona-Reina, G. EPJ Web of Conferences 140, 09039, Powders and Grains, 2017

Inmersed non-uniform test. Comparison with experimental data

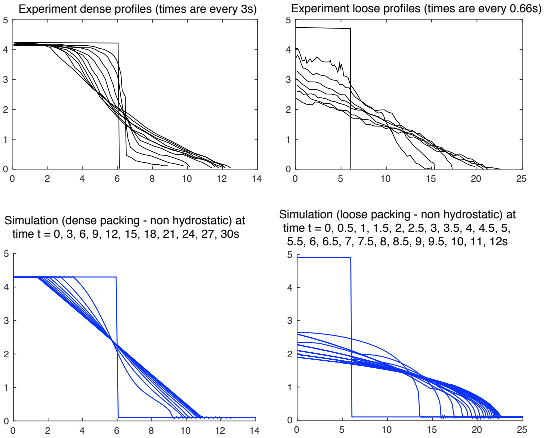
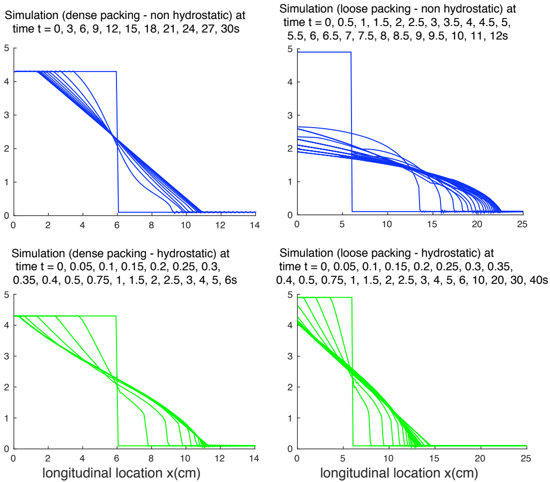


Figure: Granular mass profiles from experiments [top] and simulations (non hydrostatic [middle] and hydrostatic [bottom] pressures). left: dense initial packing ($\varphi_i = 0.6$, $L_i = 6$ cm, $H_i = 4.2$ cm); right: loose initial packing ($\varphi_i = 0.55$, $L_i = 6$ cm, $H_i = 4.8$ cm)

Inmersed non-uniform test. Comparison with experimental data



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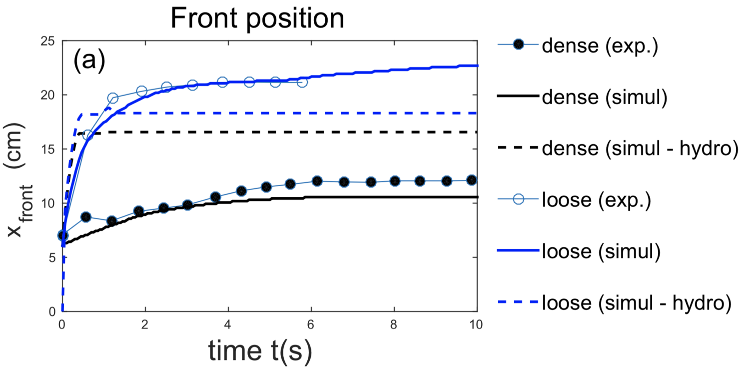

Figure: Time evolution of the front position for both initial packing from experiments (dense -●- and loose -○-) and simulations (non hydrostatic [dense –, loose –] and hydrostatic [dense – –, loose – –] pressures)

Inmersed non-uniform test. Comparison with experimental data

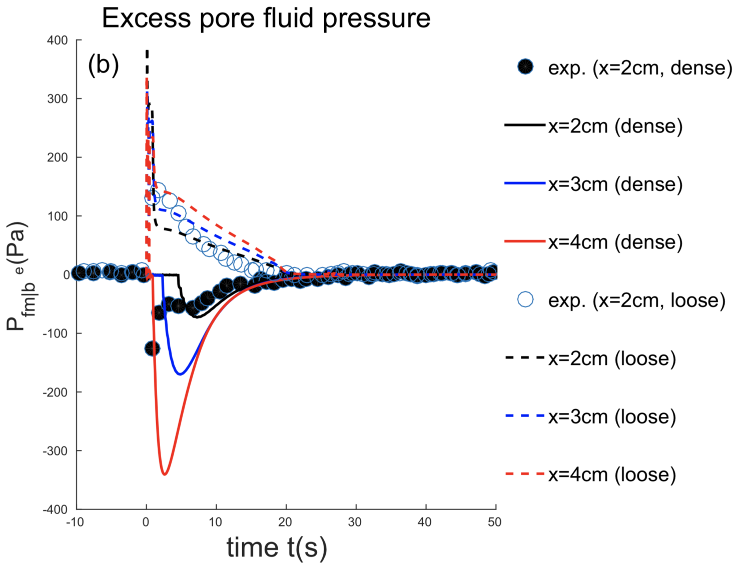

Figure: Time evolution of the pore pressure below the column from experiments (dense ● and