On the Jacques Louis Lions' multiplier method and the observability of water waves

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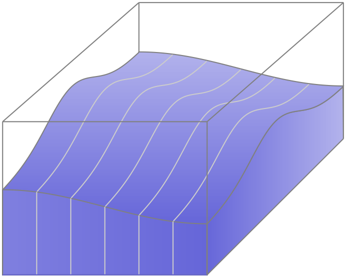
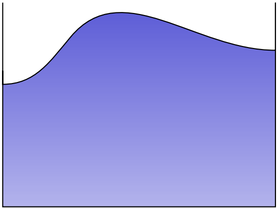
Question: **Generation** and **absorption** of water waves in **wave tanks**





Control theory for the water-wave equations

The aim of control theory is to study the possibility of forcing a system into a particular state by means of an appropriate control function.

There are many results for equations describing water waves :

- Benjamin-Ono, KdV, Saint-Venant;

see works by Cerpa, Crépeau, Coron, Dubois, Glass, Guerrero, Laurent, Linares, Ortega, Petit, Rosier, Rouchon, Russell, Zhang....

Here we consider the incompressible Euler equation with free surface.

Main differences :

- the equation is **quasi-linear** instead of semi-linear,
- the domain has a free boundary and the problem is **non local**.

Non perturbative problem $\longrightarrow$ we need a very robust method.

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EXACT CONTROLLABILITY, STABILIZATION AND PERTURBATIONS FOR DISTRIBUTED SYSTEMS *

J. L. LIONS†

The multiplier method

To find conformal transformations, Riemann studied the Laplace equation

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = f.$$

For any function v satisfying $v|_{\partial\Omega} = f$, the Green's identity implies that:

$$0 = \int_\Omega (\Delta u)(u - v)\,\mathrm{d}x = -\int_\Omega \nabla u \cdot \nabla(u - v)\,\mathrm{d}x$$

so

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x \le \left(\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)^{1/2} \left(\int_\Omega |\nabla v|^2\,\mathrm{d}x\right)^{1/2},$$

hence

$$\|\nabla u\|_{L^2}^2 \le \|\nabla v\|_{L^2}^2.$$

Reduce the existence theory to a minimization problem.

The general principle is to use the **Energy method**. This is the key to prove the existence/uniqueness of solutions: Riesz-Fréchet or Lax–Milgram theorems allow to consider equation with variable coefficients

$$\operatorname{div}(A(x)\nabla u) = 0.$$

The general principle is to use the **Energy method**. This is the key to prove the existence/uniqueness of solutions: Riesz-Fréchet or Lax–Milgram theorems allow to consider equation with variable coefficients

$$\mathrm{div}(A(x)\nabla u) = 0.$$

Well chosen multipliers allow to prove qualitative properties of the solutions. One key example : the Caccioppoli inequality, using the multiplier $\chi^2 u$ for some χ with compact support.

Used by De Giorgi to prove that *weak solutions are always Hölder continuous*.

Much more subtle then the Schauder's theory which is a perturbative analysis.

Two other fundamental multipliers:

The Pohozaev identity states that

$$\int_{\mathbb{R}^n} \Delta u\Big(\frac{n}{2}u + x\cdot\nabla u\Big)\,\mathrm{d}x = -\int_{\mathbb{R}^n} |\nabla u|^2\,\mathrm{d}x.$$

Applications: Virial identities, non-existence of stationary waves...

Rellich identity: multiply $\Delta u = 0$ by $x\cdot\nabla u$ in a Lipschitz domain $\Omega\subset\mathbb{R}^2$:

$$\int_\Gamma (x\cdot n)\big((\partial_n u)^2 - |\nabla_\Gamma u|^2\big)d\sigma + \int_\Gamma 2(\partial_n u)(x\cdot\nabla_\Gamma u)d\sigma = 0.$$

Useful if $x\cdot n>0$ on Γ: star shaped domain.

Application: Dirichlet problem for boundary data in L^2.

The **energy method** is used to prove the existence of solutions to evolution equations (Hille–Yosida, Leray). Multipliers give here also **qualitative properties**.
Consider the nonlinear Klein-Gordon equation:

$$\partial_t^2 u - \Delta u + u + u^4 = 0, \qquad x \in \mathbb{R}^3.$$

Introduce the local energy

$$E(u;\Omega,t) = \int_\Omega \Big(\frac{1}{2}(\partial_t u)^2 + |\nabla u|^2 + \frac{1}{2} m u^2 + \frac{u^5}{5}\Big)\,\mathrm{d}x.$$

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Thm. (Morawetz, 1968) $\displaystyle\int_0^T E(u;\Omega,t)\,\mathrm{d}t \lesssim E(u;\mathbb{R}^3,0)\,.$

The basic identity† that is used is

$$\begin{aligned}2r^{-1}(x\,.\,\nabla u + u)\,(u_{tt} - \Delta u + Q'(u)) &= 2(r^{-1}(x\,.\,\nabla u + u)\,u_t)_t \\ &+ \operatorname{div}\{r^{-1}(-u_t^2 x - 2(x\,.\,\nabla u)\,\nabla u + |\nabla u|^2 x - 2u\nabla u - r^{-2}u^2 x + 2Qx)\} \\ &- 2r^{-3}(x\,.\,\nabla u)^2 + 2r^{-1}|\nabla u|^2 + 2r^{-1}(uQ' - 2Q)\end{aligned}$$

with $\qquad Q(u) = \frac{1}{2}mu^2 + P(u)$

and $|x| = r$.

J.-L. Lions' multiplier method

Consider the 1D wave eq with Dirichlet boundary condition:

$$\partial_t^2 u - \partial_x^2 u = 0, \quad u(t,0) = u(t,1) = 0.$$

Multiply the equation by $x\partial_x u$ and integrate by parts in space and time:

$$\int_0^T (\partial_x u(t,1))^2 \, \mathrm{d}t = 2\int_0^1 (\partial_t u)(x\partial_x u) \, \mathrm{d}x \Big|_0^T + \iint_S \left[(\partial_t u)^2 + (\partial_x u)^2\right] \mathrm{d}x \, \mathrm{d}t$$

where $S = (0,T) \times (0,1)$, so

$$\int_0^T (\partial_x u(t,1))^2 \, \mathrm{d}t \geq (T-2)E \quad \text{where} \quad E = \int_0^1 \left[(\partial_t u)^2 + (\partial_x u)^2\right](0,x) \, \mathrm{d}x.$$

GCC: If one observes at $x = 1$ during a time $T \geq 2$, then one necessarily sees the wave; generalized by Rauch–Taylor and Bardos–Lebeau–Rauch.

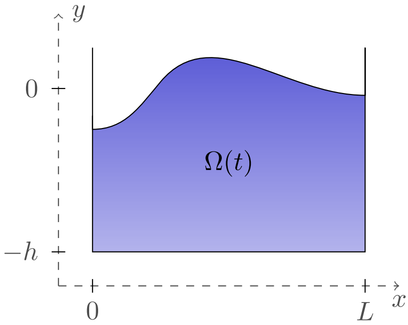
The equations

For simplicity: we consider only 2D water waves.

The fluid domain Ω has a **free surface**. At time $t \geq 0$,

$$\Omega(t) = \{\, (x, y) \in [0, L] \times \mathbb{R} \,:\, -h < y < \eta(t, x) \,\},$$

where η is an unknown.



Incompressible liquid in a domain Ω with a **free surface**. At time $t \geq 0$,

$$\Omega(t) = \{ (x,y) \in [0,L] \times \mathbb{R} \,:\, -h < y < \eta(t,x) \}.$$

The equations are

$$\begin{aligned}
&\partial_t v + v \cdot \nabla v + \nabla(P + gy) = 0 && \text{in } \Omega \\
&\operatorname{div} v = 0 && \text{in } \Omega \\
&v \cdot n = 0 && \text{on the bottom and walls} \\
&\partial_t \eta = \sqrt{1+\eta_x^2}\, v \cdot n && \text{on the free surface} \\
&P - P_{ext} = \kappa \partial_x \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right) && \text{on the free surface,}
\end{aligned}$$

g gravity, P pressure, P_{ext} external pressure, κ surface tension.

We assume that $\kappa = 1$.

[From the *Transactions of the Cambridge Philosophical Society*, Vol. VIII. p. 441.]

ON THE THEORY OF OSCILLATORY WAVES.

[Read March 1, 1847.]

$$p = g\rho y - \rho \frac{d\phi}{dt} - \frac{\rho}{2}\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right\} \quad \text{..........(1);}$$

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0 \quad \text{......................(2);}$$

$$\frac{d\phi}{dy} = 0, \text{ when } y = h \quad \text{..................(3);}$$

$$\frac{dp}{dt} + \frac{d\phi}{dx}\frac{dp}{dx} + \frac{d\phi}{dy}\frac{dp}{dy} = 0, \text{ when } p = 0 \quad \text{.......(4);}$$

Consider a potential flow (irrotationnal) so that $v = \nabla\phi$ where $\phi\colon \Omega \to \mathbb{R}$ solves

$$\Delta\phi = 0, \quad \partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + P + gy = 0$$

Then ϕ which is fully determined by $\psi(t,x) := \phi(t,x,\eta(t,x))$.

Thm (Zakharov 1968). One has

$$\frac{\partial\eta}{\partial t} = \frac{\delta\mathcal{H}}{\delta\psi}, \qquad \frac{\partial\psi}{\partial t} = -\frac{\delta\mathcal{H}}{\delta\eta} - P_{ext}$$

where $\mathcal{H}$ is the energy:

$$\mathcal{H} := \frac{1}{2}\iint_\Omega |\nabla_{x,y}\phi|^2 \,\mathrm{d}y\,\mathrm{d}x + \frac{g}{2}\int \eta^2 \,\mathrm{d}x + \kappa \int \frac{\eta_x^2}{1+\sqrt{1+\eta_x^2}}\, dx.$$

STABILITY OF PERIODIC WAVES OF FINITE AMPLITUDE ON THE SURFACE OF A DEEP FLUID

V. E. Zakharov

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 9, No. 2, pp. 86–94, 1968

variables. By introducing canonical variables, we can consider the problem of the stability of surface waves as part of the more general problem of nonlinear waves in media with dispersion [3, 4]. The re-

Pionneering works:

Zakharov, Nalimov, Yoshihara, Craig

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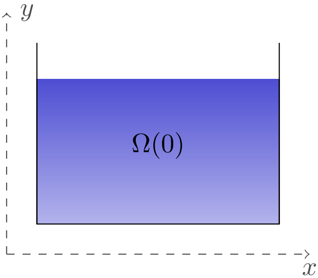
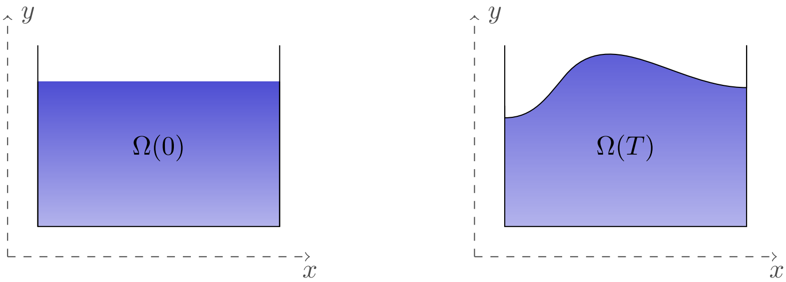
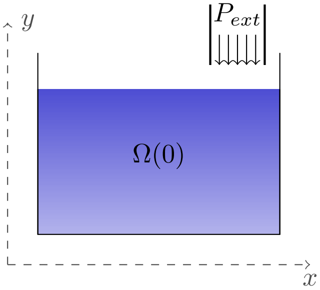
see also:



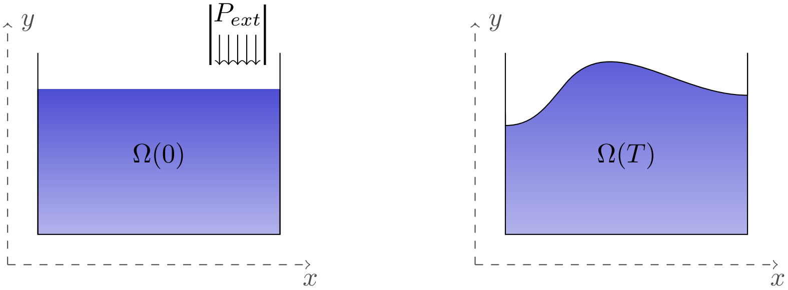
y
Ω(0)
x



y
$\Omega(0)$
x
y
$\Omega(T)$
x



y
P_{ext}
$\Omega(0)$
x
y
$\Omega(T)$
x



Theorem (A, Baldi and Han-Kwan). Given

- a time $T > 0$,
- an initial state (η_{in}, ψ_{in}) and a final state $(\eta_{final}, \psi_{final})$, small enough and smooth enough,
- a domain $\omega = (a, b)$,

there is $P_{ext}(t, x)$ supported in $[0, T] \times \omega$ such that the unique solution with initial data $(\eta, \psi) = (\eta_{in}, \psi_{in})$ satisfies $(\eta, \psi)|_{t=T} = (\eta_{final}, \psi_{final})$.

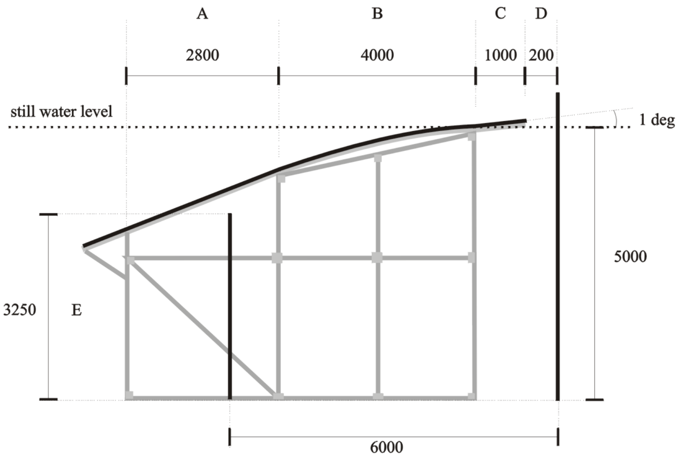
Hui Zhu extended this result to 3D water waves.
Based on the HUM method introduced by J. L. Lions.

After wave generation, **wave absorption** is the most important mechanism in a wave tank.

One wants to study the propagation in unbounded domains (open sea).

But numerical or experimental studies requires to work in a bounded domain.

–> Need to damp outgoing waves in an absorbing zone surrounding the boundaries.

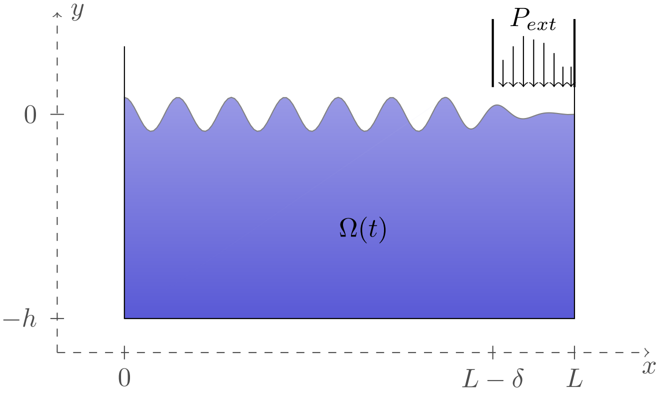


Experimental wave absorbers are passive absorbers: they consist of a beach with a mild slope.

When arriving to the artificial beach: steepening of the forward face of waves and then overturning dissipates energy.

Stabilization

Classical numerical absorbers in numerical wave tanks: Absorption of water waves in sponge boundary layer near $x = L$, by means of an external counteracting pressure produced by blowing above the free surface.



Denote by $\mathcal{H}(t)$ the energy of the fluid at time t:

$$\mathcal{H}(t) = \underbrace{\frac{g}{2}\int_0^L \eta^2 \,\mathrm{d}x + \kappa \int_0^L \left(\sqrt{1+\eta_x^2} - 1\right) \mathrm{d}x}_{\text{potential energy}} + \underbrace{\frac{1}{2}\iint_{\Omega(t)} \left|\nabla_{x,y}\phi\right|^2 \mathrm{d}y\,\mathrm{d}x}_{\text{kinetic energy}}.$$

Goal : find P_{ext} such that
(i) one has $\operatorname{supp} P_{ext}(t,\cdot) \subset [L-\delta, L]$;
(ii) $\mathcal{H}$ decays exponentially to 0.

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(i) one has supp $P_{ext}(t,\cdot) \subset [L-\delta, L]$;
(ii) $\mathcal{H}$ decays exponentially to 0.

But WW might blow up in finite time.... Here exponential decay means the following: find P_{ext} such that

$$\mathcal{H}(T) \leq \frac{C}{T}\mathcal{H}(0).$$

$$\underbrace{\mathcal{H}(T) \leq \frac{C}{T}\mathcal{H}(0)}_{\text{damping for } T > C} \quad \Rightarrow \quad \underbrace{\mathcal{H}(nT) \leq \left(\frac{C}{T}\right)^n \mathcal{H}(0)}_{\text{exponential decay}}.$$

First question: how to force the energy to decay ?

Hamiltonian damping: since

$$\frac{\partial \eta}{\partial t} = \frac{\delta \mathcal{H}}{\delta \psi}, \qquad \frac{\partial \psi}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \eta} - P_{ext}$$

we have

$$\frac{d\mathcal{H}}{dt} = \int \left[\frac{\delta \mathcal{H}}{\delta \eta} \frac{\partial \eta}{\partial t} + \frac{\delta \mathcal{H}}{\delta \psi} \frac{\partial \psi}{\partial t} \right] \mathrm{d}x = -\int \frac{\partial \eta}{\partial t} P_{ext} \, \mathrm{d}x.$$

If $P_{ext} = \chi \partial_t \eta$ with $\chi \geq 0$, then $\dfrac{d\mathcal{H}}{dt} \leq 0.$

This choice is widespread : Cao–Beck–Schultz, Clément, Grilli, Bonnefoy, Ducrozet, Baker–Meiron–Orszag,...

Theorem

Assume that

$$P_{ext}(t,x) = \chi(x)\partial_t \eta.$$

$i)$ *There exist two positive constants* δ, C *, depending explicitely on* g, κ, L *, such that, if*

$$\|\eta\|_{W^{2,\infty}} \leq \delta,$$

and if the solution exists on the time interval $[0,T]$ *, then*

$$\mathcal{H}(T) \leq \frac{C}{T}\mathcal{H}(0).$$

$ii)$ *There exists a constant* c_* *such that, if*

$$\|\eta_0\|_{H^{7/2}} + \|\psi_0\|_{H^3} \leq \varepsilon,$$

then the solution exists and is $O(\varepsilon)$ *on a time interval of size* c_*/ε *.*

Damping (decreasing energy) is easy but stabilization (exp decay to 0) is more difficult (since the equation is **quasi-linear** and **nonlocal**).

Our goal is to find P_{ext} such that (i) supp $P_{ext}(t,\cdot) \subset [L-\delta, L]$ and

$$(ii) \quad \mathcal{H} \text{ is decreasing} \quad \text{and} \quad (iii) \quad \int_0^T \mathcal{H}(t)\,\mathrm{d}t \leq C\mathcal{H}(0).$$

Then

$$\underbrace{\mathcal{H}(T) \leq \frac{1}{T}\int_0^T \mathcal{H}(t)\,\mathrm{d}t \leq \frac{C}{T}\mathcal{H}(0)}_{\text{damping for } T>C}.$$

First problem: compute

$$\int_0^T \mathcal{H}(t)\,\mathrm{d}t.$$

Lemma (An exact identity)

Consider $m \in C^\infty([0,L])$ *such that* $m(0) = m(L) = 0$. *Set*

$$\zeta = \partial_x(m\eta) - \frac{1}{4}\eta + \frac{1-m_x}{2}\eta, \quad \rho = (m-x)\eta_x + \left(\frac{5}{4} + \frac{m_x}{2}\right)\eta.$$

Then, for smooth enough solutions of the gravity water-wave equations, defined for $t \in [0,T]$,

$$\begin{aligned}\frac{1}{2}\int_0^T \mathcal{H}(t)\,\mathrm{d}t + \mathcal{Q} &= \iint P_{ext}\,\zeta\;\mathrm{d}x\,\mathrm{d}t - \int \zeta\psi\,\mathrm{d}x\,\Big|_0^T \\ &\quad + \iint \left(\frac{1-m_x}{2}\psi + (x-m)\psi_x\right) G(\eta)\psi\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \iiint \rho_x\,\phi_x\,\phi_y\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where

$$\mathcal{Q} = \int_0^T \int_0^L \left(\frac{h}{2} + \frac{\rho}{2}\right)\phi_x^2(t,x,-h)\,\mathrm{d}x\,\mathrm{d}t + \frac{L}{2}\int_0^T \int_{-h}^{\eta(t,L)} \phi_y^2(t,L,y)\,\mathrm{d}y\,\mathrm{d}t.$$

First tool: **Morawetz-Lions multiplier method** used as follows.

Set

$$\mathcal{M}(t) = \int_0^L \int_{-h}^{\eta(t,x)} \phi_x(t,x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M} = 0.$$

In fact

$$\mathcal{M} = \int_0^L I(t,x)\,\mathrm{d}x \quad \text{with} \quad I(t,x) = \int_{-h}^{\eta(t,x)} \phi_x(t,x,y)\,\mathrm{d}y.$$

Then

$$\partial_t I + \partial_x S = 0,$$

with

$$S(t,x) := \int_{-h}^{\eta} (P + \phi_x^2)\,\mathrm{d}y.$$

Multiply by m and integrate by parts (exploiting $P \geq 0$).

Second tool: a **Pohozaev** identity.

Next, we split

$$\int (G(\eta)\psi) m\partial_x\psi \,\mathrm{d}x = \int (G(\eta)\psi) x\partial_x\psi \,\mathrm{d}x + \int (G(\eta)\psi)(m-x)\partial_x\psi \,\mathrm{d}x.$$

We have a Pohozaev identity for the Dirichlet to Neumann operator:

$$\int (\partial_n\phi)(x\partial_x\psi)\, d\sigma = \Gamma + \int (\eta - x\eta_x)\left(\phi_x^2 - \phi_y^2 + 2\phi_x\,\phi_y\eta_x\right)\Big|_{y=\eta} \mathrm{d}x$$

where $\Gamma = \Gamma(t)$ is a positive term given by

$$\Gamma(t) = \frac{h}{2}\int \phi_x^2(t,x,-h)\,\mathrm{d}x + \frac{L}{2}\int_{-h}^{\eta(t,L)} \phi_y^2(t,L,y)\,\mathrm{d}y.$$

Third tool: a **Rellich** identity.

All the nonlinear terms are of the form:

$$\int \rho \left(\phi_x^2 - \phi_y^2 + 2\phi_x \, \phi_y \eta_x\right) \, \big|_{y=\eta} \, \mathrm{d}x.$$

The energy controls $\iint |\nabla_{x,y}\phi|^2 \, \mathrm{d}y \, \mathrm{d}x$, not $\int |\nabla_{x,y}\phi|^2 \, \big|_{y=\eta} \, \mathrm{d}x$.

We use the Rellich type identity:

$$\int \rho \left(\phi_x^2 - \phi_y^2 + 2\phi_x \, \phi_y \eta_x\right) \, \big|_{y=\eta} \, \mathrm{d}x = -\iint \rho_x \, \phi_x \, \phi_y \, \mathrm{d}y \, \mathrm{d}x + \frac{1}{2} \int \rho \phi_x^2 |_{y=-h} \, \mathrm{d}x,$$

which relies on

$$\partial_y \left(\rho \phi_y^2 - \rho \phi_x^2\right) + 2\partial_x \left(\rho \phi_x \phi_y\right) = 2\rho_x \phi_x \phi_y.$$

Thank you!